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Slow oscillations in an ocean of varying depth
Part 2. Islands and seamounts

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We consider slow oscillations trapped about axisymmetric islands and seamounts. ω is in the range $\lesssim \delta/2$ (ω is the frequency divided by f, the Coriolis parameter, and δ the fractional change in depth). The periods, for example, are $\gtrsim 2{\cdot}4$ days for an island with a sloping 'skirt', $h \propto r^{\frac{1}{2}}$, where h is the depth and (r, θ) are polar co-ordinates in the plane tangent to the mean sea surface. Energy leaks slowly away from the topography in Rossby waves. In the limiting case of a cylindrical island with vertical walls there are no such trapped motions, but incident Rossby waves are scattered anisotropically. If γ, the ratio of the island radius, a, to the Rossby wavelength, is small, the scattering cross-section $\sim \gamma^3 a$. The free oscillations at seamounts and islands with skirts allow much stronger scattering (with cross-section $\sim a/\gamma$, $\sim$ a wavelength), when one of their frequencies is near that of the incident wave.

The theory suggests that measurements of Rossby waves will be possible at small islands, but that the many local oscillations in the same frequency range will add some confusion.

1. Introduction

Slow oceanic oscillations with bottom topography were considered by the author (1969, hereafter referred to as I). The β-effect and the stretching of vortex lines due to motion through changes in depth were both included, with the restriction that the radius of curvature of the depth contours be much larger than the scale of the waves. Here we consider the other extreme, where the radius of curvature is small. Solutions appropriate to islands and seamounts are found, and matched to Rossby waves in a constant depth exterior.

2. Free oscillations

A stream function for the horizontal mass-flux, defined by

$$\hat{\mathbf{k}} \wedge \nabla\bar{\psi} = h\mathbf{u},$$

satisfies the vorticity equation, derived in I:

$$\nabla . \left(\frac{1}{h}\nabla\bar{\psi}_t\right) + \nabla\bar{\psi} \wedge \nabla\frac{f}{h} . \hat{\mathbf{k}} = 0,$$

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$\mathbf{\hat{k}}$ is a vertical (radial) unit vector, $\mathbf{u}$ the horizontal velocity, t the time and h the depth.

Over topography $h = h(r)$ restricted to a small region, $0 \leqslant r \leqslant a$, we may neglect variations of f and find an equation correct to $O(\gamma)$ ($\gamma \equiv a\beta/\omega f_0$ is the ratio of a to a typical Rossby wavelength),

$$\nabla^2\psi - \frac{h_r}{h}\left(\psi_r + \frac{1}{i\omega r}\psi_\theta\right) = 0, \tag{2.1}$$

where $\overline{\psi} = \psi \exp(-i\omega f_0 t)$, $f = f_0 + \beta y$. x and y are Cartesian co-ordinates directed east and north, respectively.

When $\dfrac{h_r}{rh}(\gtrless 0)$ on the average we may expect wave-like solutions with phase moving $\begin{pmatrix}\text{clockwise}\\ \text{counter-clockwise}\end{pmatrix}$ in the northern hemisphere, just as the phase velocity of Rossby waves is always to the west.

A seamount

Topography of the form

$$\begin{aligned} h &= H\exp\{\delta(r/a)^2\} \quad (r \leqslant a) \quad \text{(subscript 1)} \\ &= H\exp(\delta) \qquad\quad (r > a) \quad \text{(subscript 2)} \end{aligned}$$

is taken to model a seamount. It is paraboloidal near the origin, but slopes more steeply as the r becomes large. Equation (2.1) becomes

$$\hat{\psi}_{1,rr} + \left(\frac{1}{r} - \frac{2\delta r}{a^2}\right)\hat{\psi}_{1,r} - \left(\frac{s^2}{r^2} + \frac{2\delta s}{\omega a^2}\right)\hat{\psi}_1 = 0 \quad (r \leqslant a),$$

$$\hat{\psi}_1(r) = \psi_1 \exp\{-is\theta\}.$$

A change of the dependent variable to

$$\phi_1 = \hat{\psi}_1 h^{-\frac{1}{2}}$$

moves the third term to the zeroth derivative

$$\phi_{1,rr} + \frac{1}{r}\phi_{1,r} + \left[-\frac{s^2}{r^2} + \frac{\delta}{a^2}\left(-\frac{2s}{\omega} + 2 - \delta\frac{r^2}{a^2}\right)\right]\phi_1 = 0.$$

With error of order $\delta\omega/2s$ the last term may be neglected, leaving a Bessel equation. If the depth changes by a factor of 10 from $r = 0$ to $r = a$, δ is less than 2·5, and this is a good approximation even then, for the lower frequency modes. The free surface has been taken to be rigid, since f^2a^2/gh is very small as long as γ is.

The interior solution is

$$\overline{\psi}_1 = (H\exp\{\delta(r/a)^2\})^{\frac{1}{2}}\exp\{i(s\theta - \omega f_0 t)\}\phi_1(r),$$

$$\phi_1 = \sum_0^\infty c_{-s}\exp\{-is\theta\}J_s(\alpha r) + \sum_1^\infty c_s \exp\{is\theta\} I_s(\alpha r),$$

where $\alpha(s)$ implies the dispersion relation

$$\omega = \frac{-2\delta s}{(\alpha a)^2 - 2\delta} \quad (s < 0).\dagger$$

The phase at a of waves of given ω, s and δ is independent of a, which will simplify the matching procedure.

For $r > a$, the solutions are Rossby waves:

$$\hat{\psi}_2 = \exp\{-i\kappa x\} \sum_{-\infty}^{\infty} \exp\{is\theta\} D_s H_s^{(2)}(\kappa r) \quad (\kappa = r/a.)$$

This Hankel function represents a radially outward propagation of energy, although the phase moves inwards; as $\kappa r \to \infty$

$$H_s^{(2)}(\kappa r) \sim -(2/\pi\kappa r)^{\frac{1}{2}} \exp\{-i\kappa r + \tfrac{1}{2}i\pi(s+\tfrac{1}{2})\}.$$

Unless there is an outer boundary, motion near the origin must, therefore, decay with time to conserve energy.‡

At the discontinuity in slope the horizontal velocities of the two solutions are set equal, as in I. That is

$$\begin{aligned} \hat{\psi}_1 &\equiv (H\exp\{\delta\})^{\frac{1}{2}} \phi_1 = \hat{\psi}_2, \\ \hat{\psi}_{1\,r} &\equiv (H\exp\{\delta\})^{\frac{1}{2}} [\phi_{1,r} + (\delta/a)\,\phi_1] = \hat{\psi}_{2,r} \\ &= \frac{\partial}{\partial r}\left[\exp\{-i\kappa x\} \sum_{-\infty}^{\infty} D_s \exp\{is\theta\} H_s^{(2)}(\kappa r)\right], \end{aligned} \quad (2.2)$$

at $r = a$.

The neglect of β in the interior causes errors of order γ there. No inconsistency arises in retaining β outside, where it has room to act; for $\gamma \ll 1$ the outer solutions behave near $r = a$ like the same harmonic functions, $r^{-|s|}\exp\{is\theta\}$, as if β were identically zero. To this order, then, the slow decay of the solutions is unimportant, the outer solutions having become very small by the radius at which they become oscillatory.

Clearly the right-hand sides of (2.2) are not in separable form; each $\exp\{is\theta\}$ term is multiplied by all the D_s's. If, however, the velocities are of the same or increasing order in γ, with increasing s, the leaking that β induces between the modes is small. Keeping just two terms of $\exp\{-i\kappa x\}$ for example,

$$\exp\{-i\kappa x\}\,|_{r=a} \approx 1 - i\gamma\cos\theta.$$

Then $\hat{\psi}_2$ is

$$\left[-\frac{2i}{\pi}(\ln\gamma - 0{\cdot}12)\,D_0 + \pi(D_1 - D_{-1}) + \ldots\right]$$
$$+\exp\{i\theta\}\left[\frac{2i}{\pi\gamma}D_1 - \frac{\gamma}{\pi}(\ln\gamma - 0{\cdot}12)\,D_0 + \frac{2}{\pi\gamma}D_2 + \ldots\right] + \exp\{-i\theta\}[\ldots] + \ldots.$$

† This solution, with minor modifications, describes the low frequency oscillations of fluid rotating in a cylinder with a flat bottom and free surface (the use of equation (1.2) of I improves the result).

‡ The author has benefited from a thorough analysis by Longuet-Higgins (1967) of 'almost-trapped' gravity waves over a seamount, which have some features in common with these lower frequency motions.

If
$$D_{|s|} \gg \begin{pmatrix} sD_{|s|+1} \\ (\gamma^2/s)\,D_{|s|-1} \end{pmatrix} \quad |s| > 1,$$
$$D_1 \gg \gamma^2 \ln \gamma D_0,$$
the coupling may be neglected in (2.2). Including the effect of neighbouring azimuthal components, $\hat{\psi}_{2,r}$ in (2.2) is

$$\hat{\psi}_{2,r} = -\frac{2i}{\pi a} D_0 + \exp\{i\theta\}\left(-\frac{1}{\pi a}\right)\left[\gamma \ln \gamma D_0 + \frac{2i}{\gamma} D_1 + \frac{2}{\gamma} D_2 + \ldots\right] + \ldots.$$

The sth component involves just D_s under the same conditions. The coupling will thus be weaker for the lower s-modes. D_0 is a special case, however, since the topography is ineffective when there are no radial velocities.

The equations are now the same as if the modulation $\exp\{-i\kappa x\}$ had been neglected in (2.2), except for the isotropic balance. In practice we are only interested in the first few modes.

With these conditions satisfied there will be solutions for D_s and c_s only if the determinant of the matching equations vanishes. This gives the transcendental relations

$$-\left.\frac{J_{|s|,r}(\alpha r)}{J_{|s|}(\alpha r)}\right|_{r=a} = \frac{\delta + s}{a} \quad (s < 0),$$

or

$$-\frac{J_{|s|-1}(\alpha a)}{J_{|s|}(\alpha a)} = \frac{\delta}{\alpha a}$$

and

$$\frac{I_{s-1}(\alpha a)}{I_s(\alpha a)} = -\frac{\delta}{\alpha a} \quad (s > 0).$$

For a sufficiently small seamount, $\delta \ll \alpha a$,

$$J_{|s|-1}(\alpha a) = 0 \quad (s < 0).$$

There are no roots for $s > 0$, and the isotropic mode is largely forced by $c_{\pm 1}$.

The frequency that results is plotted in figure 1 against the height of the seamount (it is *independent* of a), for several azimuthal modes. There is an almost linear increase in ω with δ, to quite high values (for a quasigeostrophic wave); the appearance of high frequencies near such topography could represent either these local oscillations or a very long Rossby wave.

As the number of radial nodes increases the frequency drops only slowly, because the mean angle at which the fluid crosses contours does not change rapidly. When circular nodes are present, however, this angle and the frequency are greatly reduced.

Viscosity and non-linearities will probably be destructive to the modes with circular nodes. There is, therefore, little reason to calculate the exact rate of inviscid decay due to radiation (the smallness of γ for the faster oscillations makes it less important for these cases also).

With Longuet-Higgins' (1967) gravity waves over a circular cylinder, the trapping mechanism is somewhat weaker than for quasigeostrophic waves, requiring a great contrast in depths for the decay rate to be small. If this same depth profile is used instead of the 'exponential paraboloid' the principal quasi-geostrophic modes merge close to the frequency

$$\omega = \frac{\delta_1}{2+\delta_1}, \quad \delta_1 \equiv 1 - \frac{h_1}{h_2},$$

where h_1 is the depth over the profile. This value approximates well the conglomeration of lower modes in the previous problem; see figure 1. It is also the same as the frequency of waves trapped along a straight step in the ocean floor.

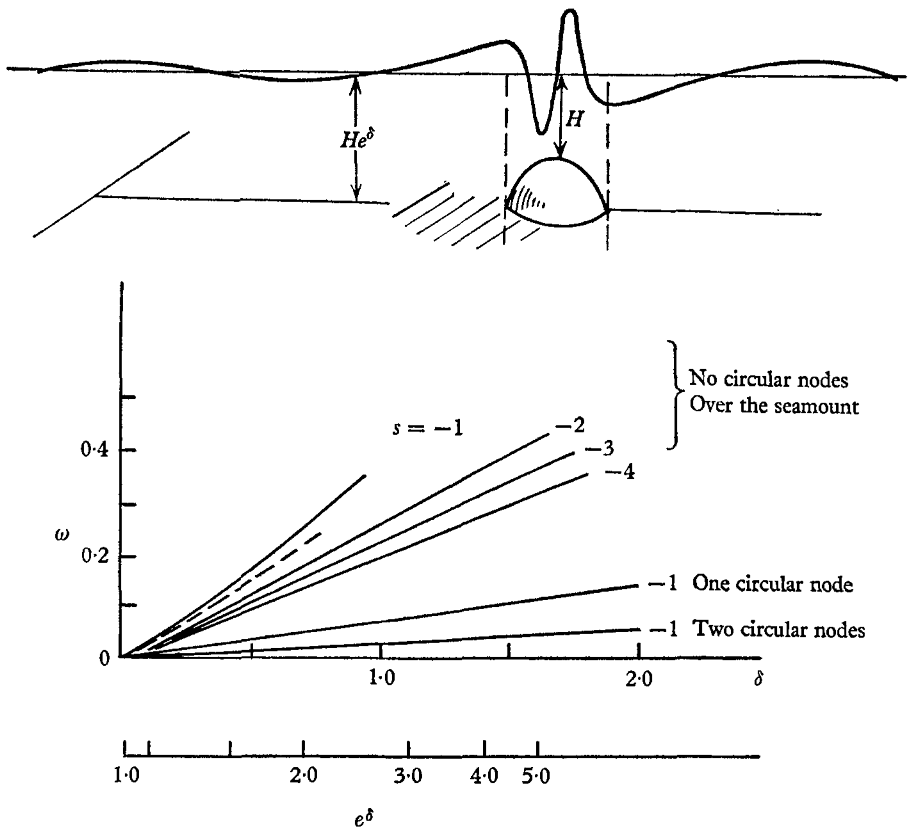

Figure 1. Form of the oscillations above a seamount, and the dependence of frequency on the height, He^{δ}. The frequency is also given for topography in the form of a circular cylinder (dashed line).

An island with a 'skirt'

Most tidal measurements are made near islands or at a coast. It has been suggested [Longuet-Higgins 1966; Rattray & Charnell 1966] that the low frequency spectral content of the records from islands may be due to Rossby waves. Following the ideas of the previous section we present an alternative possibility (for regions away from the equator) in terms of oscillations almost trapped about a symmetrical island with a sloping 'skirt'.

The analysis is almost the same as before, but with a different depth profile.

When $\gamma \equiv \kappa a_2 \ll 1$, where a_2 is the radius at which the topography intersects the uniform depth exterior, β may be neglected over the slopes. Since

$$f^2 a_2^2/gH \ll 1$$

if γ is small, the surface will be considered rigid.

For any of the family of profiles

$$h = H\left(\frac{r}{a_2}\right)^{\Sigma} \quad (r \leqslant a_2),$$

the equation

$$\nabla^2 \hat{\psi}_1 - \frac{h_r}{h}\left(\frac{\partial}{\partial r} + \frac{s}{\omega r}\right)\hat{\psi}_1 = 0, \quad \overline{\psi}_1 = \hat{\psi}_1(r)\exp\{i(s\theta - \omega f_0 t)\},$$

becomes

$$\hat{\psi}_{1,rr} + \frac{1-\Sigma}{r}\hat{\psi}_{1,r} - \left(\frac{s\Sigma}{\omega r^2} + \frac{s^2}{r^2}\right)\hat{\psi}_1 = 0. \tag{2.3}$$

This has the solution $\hat{\psi}_1 = Ar^{p_1} + A'r^{p_2}$,
where p_1, p_2 are roots of

$$p^2 - \Sigma p - \left(\frac{s\Sigma}{\omega} + s^2\right) = 0. \tag{2.4}$$

The motions are independent of the scales H and a_2.

Phillips (1965) has considered the paraboloidal member of this set, $\Sigma = 2$, which also yields the high frequency Kelvin and inertial–gravitational waves without approximation. His object was to study 'model' Rossby waves in detail. Successful experiments were performed by Ibbetson [see the above paper and Ibbetson & Phillips (1967)].

We choose $\Sigma = \frac{1}{2}$ to represent a gradual increase in depth to $r = a_2$, where the slope is discontinuous. Outside this radius the sea-bed is taken to be level, $h = H$. If the depth at $r = a_1$ is 200 m, for example, it reaches 2000 m at $r = 10a_1$.

The solutions are oscillatory in space if p is complex.

$$\hat{\psi}_1 = r^{p_r}\exp\{ip_i \ln r\} \quad (p = p_r + ip_i).$$

At the origin the dynamic effect h_r/rh becomes infinite, giving the solutions unbounded variation there. We exclude the origin, however, with a cylindrical island whose walls stand at $r = a_1$. The boundary condition at a_1 is

$$\hat{\psi}_1 = 0, \quad |s| > 0$$

(the isotropic mode has no radial velocity component, and hence is insignificant). This should be valid if the impedance to exterior motions of the shallow shelf about a real island is sufficiently great.

The interior solution is thus

$$\hat{\psi}_1 = A(r^{p_1} - a^{p_1 - p_2} r^{p_2}) \quad (a_1 \leqslant r \leqslant a_2).$$

Beyond $r = a_2$, β is included. The appropriate solution is again

$$\hat{\psi}_2 = \exp\{-i\kappa x\}\sum_{-\infty}^{\infty} D_s \exp\{is\theta\} H_s^{(2)}(\kappa r) \quad (r > a_2),$$

a Rossby wave carrying energy outwards. With $\gamma \ll 1$, $\hat{\psi}_2$ acts like a separable harmonic function near the topography:

$$\hat{\psi}_2 \approx \sum_{-\infty}^{\infty} D_s \exp\{is\theta\} \frac{i(|s|-1)!}{\pi} \left(\frac{2}{\kappa r}\right)^{|s|} \begin{Bmatrix} 1 \\ (-1)^{|s|} \end{Bmatrix} \quad \left(a_2 < r \ll \frac{1}{\kappa}\right) \quad \text{for} \quad \begin{Bmatrix} s > 0 \\ s < 0 \end{Bmatrix}.$$

The decay of the oscillations appears only at the next order in γ, so the trapping is efficient.

From the matching of $\hat{\psi}$ and $\hat{\psi}_r$ at $r = a_2$ it follows that

$$\left(\frac{a_1}{a_2}\right)^{p_1 - p_2} = \frac{|s| + p_1}{|s| + p_2}. \tag{2.5}$$

For $\Sigma = \frac{1}{2}$

$$p_1, p_2 = \tfrac{1}{4} \pm \left(\tfrac{1}{16} + s^2 + \frac{s}{2\omega}\right)^{\frac{1}{2}}.$$

With $a_1/a_2 < 1$ the choice $p_1 > p_2$ for purely real solutions of p_1, p_2 implies

$$\left(\frac{a_1}{a_2}\right)^{p_1 - p_2} < 1 \quad \text{while} \quad \frac{|s| + p_1}{|s| + p_2} > 1,$$

which contradicts (2.5). Hence $\hat{\psi}$ is always oscillatory with

$$p_1 = p_2^* = \tfrac{1}{4} + ip_i; \quad ip_i \equiv \left(\tfrac{1}{16} + s^2 + \frac{s}{2\omega}\right)^{\frac{1}{2}};$$

and $s < 0$ for non-trivial solutions: the phase moves clockwise. The solution over the skirt is the sum of an incoming and an outgoing wave,

$$\hat{\psi}_1 = Ar^{\frac{1}{4}} \sin(p_i \ln(r/a_1)). \tag{2.6}$$

(The incoming wave does not require an energy source at ∞; merely a reflexion at $r = a_2$.)

The condition (2.5) may now be written

$$\tan \xi = \frac{-\xi}{(|s| + \frac{1}{4}) \ln(a_2/a_1)},$$

where

$$\xi = p_i \ln(a_2/a_1).$$

This is a simple transcendental equation for p_i and hence the frequency. The roots are plotted for $s = -1, -2$ in figure 2 on the dispersion curve

$$\omega = \frac{-\frac{1}{2}s}{p_i^2 + s^2 + \frac{1}{16}},$$

which is similar in form to that of plane Rossby waves; p_i acts as a radial wave-number. For given a_2/a_1 each successively lower frequency mode gains one more node about the island. The highest frequencies arise when $\hat{\psi}$ has no such nodes; for $s = -1$ this is

$\omega = 0{\cdot}41$, for $a_2/a_1 = 10^3$ (a period of 2·4 days),

$\omega = 0{\cdot}36$, for $a_2/a_1 = 10^2$ (a period of 2·8 days),

$\omega = 0{\cdot}23$, for $a_2/a_1 = 10$ (a period of 4·3 days), at latitude 30°.

The various azimuthal modes are more greatly separated in frequency than in the case of a seamount.

The form of $\hat{\psi}$ is shown in figure 3 for $s = -1$ and two circular nodes. The free-surface displacement, η, is given by (2.6) with $r^{\frac{1}{2}}$ replaced by $r^{-\frac{1}{2}}$, and hence the

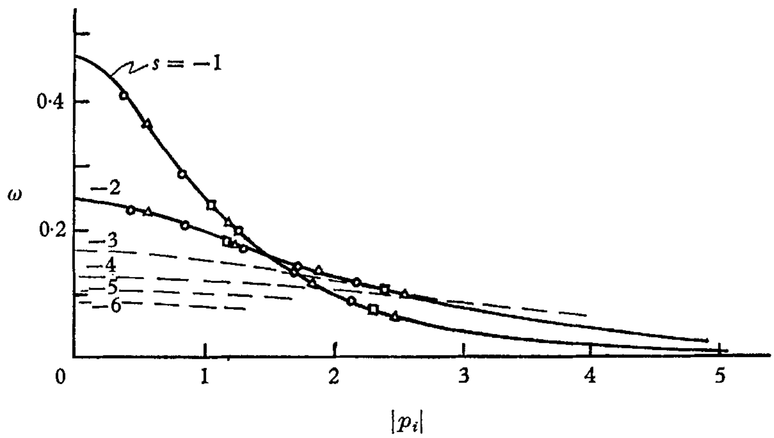

FIGURE 2. Dispersion curves for waves about an island with a skirt. a_1 and a_2 are the radii of the cylindrical island, and the outer edge of the skirt, respectively. ○, $a_2/a_1 = 10^3$; △, $a_2/a_1 = 10^2$; □, $a_2/a_1 = 10$.

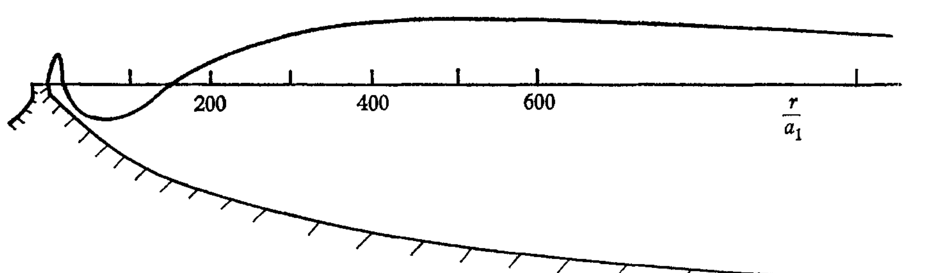

FIGURE 3. Form of the stream function for $s = -1$, two circular nodes. The velocities are greatest near the island.

'run-up' of a single mode amplifies η by a factor of order $(a_2/a_1)^{\frac{1}{2}}$ over its value at $r = a_2$. The circumferential velocity, u, is a factor $\sim (a_2/a_1)^{\frac{3}{2}}$ more intense near the island; this could be quite considerable. For other profiles $h = H(r/a_2)^{\Sigma}$ the run-up enhances η and u by $\sim (a_2/a_1)^{\frac{1}{2}\Sigma}$ and $\sim (a_2/a_1)^{1+\frac{1}{2}\Sigma}$ respectively.

Practical considerations favour the appearance of the lower modes; they will be the most responsive to large-scale forcing effects, the least subject to distortion by irregularities in the topography, and the least disturbed by non-linearities and friction.

3. Scattering of Rossby waves

Cylindrical island

Just as a ridge tends to reflect an incident Rossby wave, an isolated seamount or island in a field of waves will produce scattering, and the motion over the slopes can be described by the eigenfunctions just calculated.

To demonstrate some of the peculiarities that the anisotropy due to β gives the problem, we first treat a simpler situation: the scattering due to an island in the form of a right circular cylinder (without a skirt).

The equation is that of simple Rossby waves,

$$(\nabla^2+\kappa^2)\hat{\phi}_2 = 0, \quad \overline{\psi}_2 = \phi_2 \exp\{-i(\kappa x+\omega f_0 t)\}, \tag{3.1}$$

with the spatial dependence obeying

$$\overline{\psi} = A, \quad \text{a constant,}$$

on $r = a$; that is,

$$\hat{\phi}_2 = A\exp\{i\kappa a\cos\theta\} \quad (r = a). \tag{3.2}$$

If we were to set A equal to zero the problem would be identical to that of electromagnetic waves incident on a conducting cylinder, when the electric vector is parallel to the cylinder axis. If the radius were much smaller than a wavelength the scattering would be isotropic, yet even as $\gamma\to 0$ it would be quite strong. For acoustic waves in the presence of a rigid cylinder ($\phi_{2,r}|_{r=a} = 0$) the incident field adjusts more easily to the boundary. The scattering for small radius is weaker, and disappears with γ. The isotropic and $\cos\theta$ components dominate.

To determine A we note that the field due to an oscillating, vortex-like motion centred on the origin would not disturb either the inviscid boundary condition or the condition at infinity, yet it is disallowed by Kelvin's theorem. The circulation about the contour $r = a$ cannot vary with time, since this would imply a many-valued pressure field at $r = a$. We thus require also that

$$\int_0^{2\pi} \overline{\psi}_{2,r}|_{r=a}\, d\theta = 0. \tag{3.3}$$

The plane wave, incident from infinity at an angle $\frac{1}{2}(\theta_0+\pi)$ from east, has an expanded form appropriate to a circular boundary:

$$\overline{\psi}_i = \exp\{-i\kappa x\}\exp\{i\kappa(y\sin\theta_0+x\cos\theta_0)-\omega f_0 t\}$$

or

$$\hat{\phi}_i = \sum_{-\infty}^{\infty} i^s \exp\{is(\theta-\theta_0)\}J_s(\kappa r)$$

(the transformation in (3.1), introduced by Longuet-Higgins, moves the circular wave-number locus until it is centred on the origin).

The scattered wave $\hat{\phi}_s$ is the solution of (3.1) whose energy moves radially outwards:

$$\hat{\phi}_s = \sum_{-\infty}^{\infty} D_s \exp\{is\theta\}H_s^{(2)}(\kappa r).$$

The first boundary condition requires that

$$\sum_{-\infty}^{\infty} \exp\{is\theta\}(i^s J_s(\kappa a)\exp\{-is\theta_0\} + D_s H_s^{(2)}(\kappa a)) = A = \sum_{-\infty}^{\infty} i^s \exp\{is\theta\} J_s(\kappa a),$$

or

$$D_s = (A - \exp\{-is\theta_0\}) i^s (J_s/H_s^{(2)}) \quad (J_s \equiv J_s(\kappa a)). \tag{3.4}$$

A is determined by the second boundary condition:

$$\begin{aligned} 0 &= \int_0^{2\pi} \psi_{2,r}|_{r=a}\, d\theta \\ &= \int_0^{2\pi} (\exp\{-i\kappa x\}\phi_2)_r|_{r=a}\, d\theta \\ &= \int_0^{2\pi} \exp\{-i\kappa x\}\phi_{2,r}|_{r=a}\, d\theta \end{aligned}$$

by (3.2). The terms independent of θ in

$$[\Sigma \exp\{is\theta\}(-i)^s J_s][\Sigma \exp\{is\theta\}(i^s J_{s,r}\exp\{-is\theta_0\} + D_s H_{s,r}^{(2)})]_{r=a} = 0$$

must therefore sum to zero. That is,

$$\Sigma(-i)^s[J_s J_{s,r} i^s \exp\{is\theta_0\} + D_s J_s H_{s,r}^{(2)}] = 0$$

and, using (3.4),

$$A = \left.\frac{\Sigma(-1)^s i^{2s}\exp\{-is\theta_0\}(-J_s J_{s,r} + J_s^2(H_{s,r}^{(2)}/H_s^{(2)})}{\Sigma(-1)^s i^{2s}(J_s^2(H_{s,r}^{(2)}/H_s^{(2)})}\right|_{r=a}.$$

If we now restrict the size of the island to a small fraction of a wavelength,

$$\gamma \ll 1$$

the lowest *two* orders of A must be retained:

$$A \approx 1 + \gamma^2 \ln\gamma(1-\cos\theta_0) \quad (\gamma \ll 1);$$

substituting in (3.4),

$$D_0 = \frac{i\pi}{2}\gamma^2(1-\cos\theta_0)$$

$$D_s = [1-\exp\{-is\theta_0\}] i^{s-1}\frac{\pi}{|s|!(|s|-1)!}\left(\frac{\gamma}{2}\right)^{2|s|} \quad (s \neq 0).$$

The dominant modes are $s = 0, \pm 1$ giving, to $O(\gamma^2)$

$$\overline{\psi}_s = \exp\{-i(\kappa x + \omega f_0 t)\}\frac{i\pi}{2}\gamma^2[(1-\cos\theta_0)H_0^{(2)}(\kappa r) + i(\cos\theta - \cos(\theta-\theta_0)H_1^{(2)}(\kappa r)].$$

The orientation of the scattered waves is shown in figure 4.

The effect of the anisotropy shows up best if we plot scattered *energy-flux* against the angle of the observer from east, with respect to the island. At a great distance the velocities become

$$\overline{\psi}_{s,r} \to -i\kappa(1+\cos\theta)\overline{\psi}_s,$$

$$(1/r)\overline{\psi}_{s,\theta} \to i\kappa\sin\theta\,\overline{\psi}_s.$$

The largest contribution to $(1/r)\overline{\psi}_{s,\theta}$ comes from the travelling modulation, $\exp\{-i\kappa x\}$. The energy density, normalized to that of the incident wave, is

$$\frac{\frac{1}{2}(u^2+v^2)}{\kappa^2(1+\cos\overline{\theta}_0)\,|\overline{\psi}_i|^2} = \frac{1+\cos\theta}{1+\cos\overline{\theta}_0}\frac{|\overline{\psi}_s|^2}{|\overline{\psi}_i|^2}$$
$$= \frac{\pi}{2}\gamma^3\frac{a}{r}\left(\frac{1+\cos\theta}{1+\cos\overline{\theta}_0}\right)[(1+\cos\overline{\theta}_0)-(\cos\theta+\cos(\theta-\overline{\theta}_0))]^2,$$

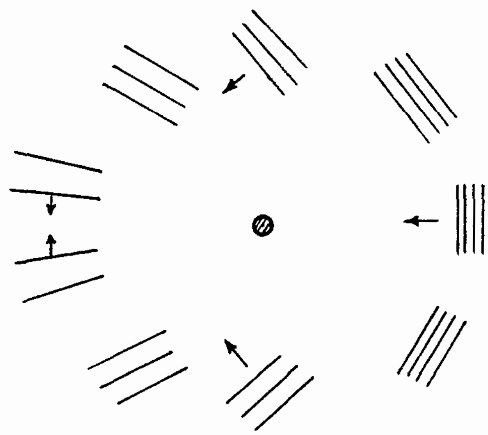

FIGURE 4. Scattered wave-crests about a circular cylinder. The phase velocity is directed along the arrows.

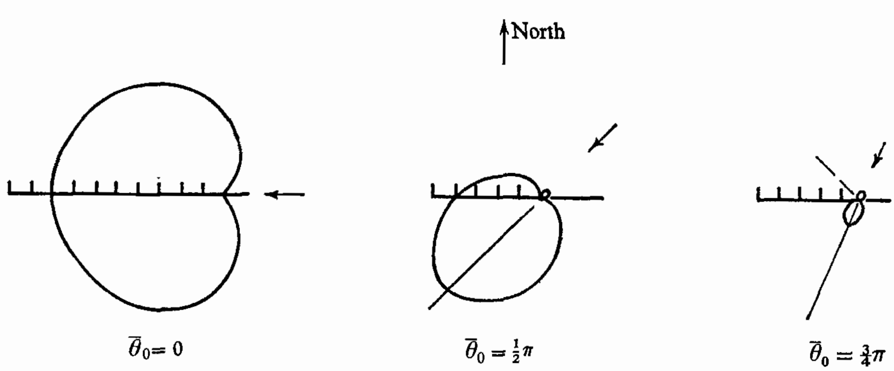

FIGURE 5. Scattered energy flux for a right circular cylinder as a function of angle. $\overline{\theta}_0$ is the angle of the incident *group* velocity. (The arrows show the incident phase velocity.)

$\overline{\theta}_0$ is the angle of the incident group velocity. The group velocity has magnitude

$$\omega f_0/\text{westward wave no.} = \frac{2\omega^2 f_0^2}{\beta(1+\cos\theta)},$$

where θ is its direction. The relative energy flux is therefore just

$$F = \frac{|\overline{\psi}_s|^2}{|\overline{\psi}_i|^2} = \frac{\pi}{2}\gamma^3\frac{a}{r}[(1+\cos\overline{\theta}_0)-(\cos\theta+\cos(\theta-\overline{\theta}_0))]^2$$

in the limit $\kappa r \to \infty$, which is plotted in figure 5. The scattering is symmetric about the direction of the incident phase, forming a weak shadow. F varies greatly with the angle of incidence, even for this simple obstacle. The energy *density* is usually concentrated to the east of the island (the short-wave region), but its flux depends also on the variation of group velocity with θ.

The total scattering cross-section, Q, is the width of a beam of the incident wave containing as much energy flux as does the whole scattered wave. In problems with homogeneous boundary conditions on the cylinder this can far exceed the physical width of the cylinder.

Here we have

$$Q = \int_0^{2\pi} rF\,d\theta$$
$$= \pi^2\gamma^3(1+(\cos\overline{\theta}_0+1)^2)\,a.$$

It is formally a small fraction of the width of the island in this limit, although the large numerical factor makes the scattering of some practical interest.

Non-linear effects would seem, at first sight, to invalidate the solution when γ is so small that columns of fluid near the island move through distances of order a. The advective terms, however, are not necessarily large then. In fact such motion tends to be irrotational, so that the change of vorticity following a particle should be well represented by the linear approximation.

Although the particle velocities are altered in the presence of the cylinder, the sea surface displacement is virtually that of the incident wave, for $\gamma \ll 1$. Since $\overline{\psi}$ is proportional to the surface elevation [to $O(\omega)$], the expression for $\overline{\psi}_s(\sim\gamma)$ shows this to be true.

Scattering by a seamount

The scattering problem for a paraboloidal seamount is similar to that just treated. This profile, however, has its own set of characteristic oscillations which, if excited, will persist for many periods. If energy is continuously beamed at the seamount near one of its eigenfrequencies, therefore, we expect unusually large amplitudes to arise. The solution is the same as that given by (2.2) but with an incident wave, of unit amplitude in $r > a$. Instead of slow decay of the solution with time, there is a steady rate of scattering of the incident energy.

We set

$$\hat{\psi}_2 = \hat{\psi}_s + \hat{\psi}_i$$

as before,

$$\hat{\psi}_i = \exp\{-i\kappa x\}\exp\{i\kappa(x\cos\theta_0+y\sin\theta_0)\}$$
$$= \sum_{-\infty}^{\infty} i^s \exp\{is(\theta-\nu_0)\}J_s(kr); \quad k \equiv -2\kappa\cos\nu_0.$$

ν_0 is the angle of incidence of ψ_i, corresponding to group velocity in the direction $2(\nu_0-\pi) \equiv \overline{\theta}_0$. The matching equations are no longer homogeneous so that there are solutions for all values of the frequency. We have

$$(H\exp\{\delta\})^{\frac{1}{2}}\hat{\phi}_1 = \hat{\psi}_s + \hat{\psi}_i,$$
$$(H\exp\{\delta\})^{\frac{1}{2}}(\hat{\phi}_{1,r} + (\delta/a)\hat{\phi}_1) = \hat{\psi}_{s,r} + \hat{\psi}_{i,r},$$

at $r = a$. With $\hat{\psi}_i$ absent the equations were separable when the seamount was small compared with a Rossby wave, and when the velocities in the different azimuthal ($\exp\{is\theta\}$) modes did not decrease in order (in γ) when $|s|$ increased above unity. At $r = a$ the scattered wave is almost cylindrical, the asymmetric modulation $\exp\{-i\kappa x\}$ having little effect. It must be retained in $\hat{\psi}_i$, however, where it contributes equally, everywhere. The topographic restoring force disappears for isotropic scattering ($s = 0$) and so this case will be treated separately.

The matching equations yield, for the interior wave amplitudes,

$$c_s = 2^{|s|+2}(-i\cos\nu_0)^{|s|+1}(\pi\alpha\Delta)^{-1}$$
$$= O(\gamma).$$

Δ is the determinant of the coefficients:

$$\Delta = -(H\exp\{\delta\})^{\frac{1}{2}}\left[\left(\alpha J_{|s|-1}+\frac{\delta-|s|}{a}J_{|s|}\right)H_s^{(2)}-J_{|s|}H_{s,r}^{(2)}\right]_{r=a},$$

$$\alpha^2 = \frac{2\delta}{a^2}\left[\frac{|s|}{\omega}+1\right],$$

where $J_s \equiv J_s(\alpha r)$, $H_s^{(2)} \equiv H_s^{(2)}(\kappa r)$, and J_s is replaced by I_s when $s > 0$.

The scattered wave amplitudes are

$$D_s = (i^s\exp\{-is\nu_0\}(H\exp\{\delta\})^{\frac{1}{2}}(-2\cos\nu_0)^{|s|})\left(\frac{\gamma}{2}\right)^{|s|}$$
$$\times\left[\alpha J_{|s|-1}+\frac{\delta-2|s|}{a}J_{|s|}\right][|s|!\Delta]^{-1}.$$

With $\gamma \ll 1$, the H_s terms are dominated by $-iY_s$. Unless there is a cancellation of these terms (as at resonance), the J_s may be neglected. The dominant scattering is then from D_0, D_{-1}, D_1:

$$D_1 = -\pi(1+\exp\{-2i\nu_0\})\left(1-2\frac{I_1}{\alpha_1 aI_0+\delta I_1}\right)\left(\frac{\gamma}{2}\right)^2,$$

$$D_{-1} = -\pi(1+\exp\{2i\nu_0\})\left(1-2\frac{J_1}{\alpha_1 aJ_0+\delta J_1}\right)\left(\frac{\gamma}{2}\right)^2,$$

D_0, c_0 will be estimated below.

As $\delta \to 0$ these amplitudes vanish, but if only $\delta/\omega \geqslant O(1)$ the scattering approaches that due to a cylindrical island. The velocities and $\overline{\psi}$ are not greatly enhanced by the topography. Although the restoring force there is large, the penetration of the wave into $r < a$ is slight. In fact, the ratio of the maximum velocity over the seamount to the typical external velocity is $\sim 1, e^{\delta}, (\delta/\omega)^{\frac{1}{2}}e^{\delta}$ for $(\delta/\omega)^{\frac{1}{2}}$ small, of order unity, and large, respectively. The 'amplification' varies almost linearly with the depth contrast.

When the incident wave has a frequency near an eigenvalue for the seamount, however, the amplitudes become much larger. The Y_s terms cancel in the expression Δ, leaving

$$\Delta_s \simeq \frac{2(He^{\delta})^{\frac{1}{2}}}{a|s-1|!}\left(\frac{\gamma}{2}\right)^{|s|}\begin{Bmatrix}J_{|s|}\\(-1)^s I_{|s|}\end{Bmatrix}\quad\begin{Bmatrix}s<0\\s>0\end{Bmatrix}.$$

$\hat{\psi}_1$ over the seamount is then $\sim \gamma^{-|s|}$, while the scattered waves are ~ 1. The total scattering cross-section is $\sim a/\gamma$ which is much greater than the width of the seamount. At resonance the separability of the problem breaks down, and the neighbouring azimuthal modes will be forced directly by the resonant motion, reducing somewhat the resonance effect.

The isotropic scattering will now be estimated. The topography provides no restoring force for this mode, and so it is dominated by the $s = \pm 1$ solutions. In the interior ψ satisfies

$$\frac{h}{r}\left(\frac{r}{h}\psi_{1,r}\right)_r = \frac{\gamma}{ia}\left(\cos\theta\psi_{1,r} + \frac{\sin\theta}{r}\psi_{1,\theta}\right) + \frac{ih}{\omega}\nabla\psi_1 \wedge \nabla\frac{1}{h}.\hat{\mathbf{k}} - \frac{\psi_{1,\theta\theta}}{r^2} \quad (r \leqslant a)$$

including the β-effect. Assuming

$$\psi_1 = \sum_{-\infty}^{\infty} \hat{\psi}^{(s)}(r)\exp\{is\theta\}$$

the isotropic balance is then

$$\frac{h}{r}\left(\frac{r}{h}\hat{\psi}_r^{(0)}\right)_r = \frac{\gamma}{2a}\left[\hat{\psi}_r^{(1)} + \psi_r^{(-1)} - \frac{1}{r}(\psi^{(1)} + \psi^{(-1)})\right] \quad (r \leqslant a),$$

relating $\hat{\psi}^{(0)}$ weakly to the neighbouring components. Taking the values already calculated for $\psi^{(\pm 1)} (\equiv h^{\frac{1}{2}}\phi^{(\pm 1)})$ we estimate

$$\psi_r^{(0)} \sim \frac{\gamma^2}{a}\left(1 + \frac{\delta}{\omega}\right)$$

$$\psi^{(0)} \sim \gamma^2\left(1 + \frac{\delta}{\omega}\right) + \text{const.}$$

The matching to the external field at $r = a$ of both $\hat{\psi}^{(0)}$ and $\hat{\psi}_r^{(0)}$ yields only a slight modification of the incident wave:

$$1 + D_0 H_0 = \psi^{(0)},$$

$$D_0 H_{0,r} = \psi_r^{(0)}.$$

The constant in $\psi^{(0)}$ is unity, and D_0 is $\leqslant O\{\gamma^2(1+\delta/\omega)\}$ so that the free surface motion over a small seamount will usually be just that of the external wave, although the velocities are greatly altered. At resonance of the $s = 1$ or -1 modes, however, $D_0 \sim 1$, and the surface is locally deformed.

Applications

In applying the results of this paper to the ocean, several deviations from reality must be accounted for. The density stratification will have an effect on the eigenfrequencies, but not on the nature of the results. In addition to the slowly leaking barotropic mode there will be a trapped baroclinic motion, on a scale

$$\left(\frac{f^2}{g(\Delta\rho/\rho)H}\right)^{-\frac{1}{2}}$$

where $\Delta\rho/\rho$ is the fractional density difference, in a two-layer model. Problems of this nature will be treated in a later paper.

The most serious aspect that has been neglected is the asymmetry of isolated topographic features. One expects intuitively, from listening to various objects being struck or dropped, that a bell (to use Longuet-Higgins' analogy) is indeed an exceptional form. The lower modes may, however, survive by averaging the irregularities beneath them, although the sharpness of the resonances should surely be reduced. Because Rossby waves are so long the necessity that the topography be isolated on a flat seabed would seem to be unrealistic. The interior motions, however, should be insensitive to 'roughness' in the exterior, as long as it is not comparable in size to δ.

The scattering due to an island with a skirt is very like that for seamount. The displacement of the free surface at the shore differs from that at sea by only $O(\gamma)$ unless the frequency is nearly resonant. The velocities, however, are strongly altered if only $\delta/\omega \gtrsim 1$ for a seamount; always for an island. Comparison with the results of I shows that Rossby waves will be much more evident near islands, than at a coast (in agreement with recent observations by Wunsch (1967)).

The resonances are dense (take a vertical section of figure 1, for example), and occur for arbitrarily small a (in a scattering problem for gravity waves, on the other hand, resonances usually appear only when the topography is as broad as an incident wavelength). The response should therefore be shown coherent between neighbouring islands before it is identified with a large-scale motion. The amplitude of the trapped oscillations will depend on the damping effects of radiation, friction, non-linear advection, and asymmetries in the topography. In spite of these uncertainties, the decay of such oscillations could be a useful measure of turbulent friction in the ocean, since radiation damping should be slight. Waves trapped over ridges and shelves might also be used.

The cumulative importance of small seamounts in a rough region like the western Pacific could be great, to Rossby waves short enough that $\bar{\delta} > \omega$, where $\bar{\delta}$ is now the root-mean-square topographic height. Some problems with continuously varying depth will be considered in a later paper.

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REFERENCES

IBBETSON, A. & PHILLIPS, N. A. 1967 Some laboratory experiments on Rossby waves in a rotating annulus. *Tellus*, **19**, 81.

LONGUET-HIGGINS, M. S. 1966 Planetary waves on a hemisphere bounded by meridians of longitude. *Phil. Trans. Roy. Soc.* A **260**, 317.

LONGUET-HIGGINS, M. S. 1967 On the trapping of wave energy round islands. *J. Fluid Mech.* **31**, 781.

PHILLIPS, N. A. 1965 Elementary Rossby waves. *Tellus*, **17**, 295.

RATTRAY, M. & CHARNELL, R. L. 1966 Quasigeostrophic free oscillations in enclosed basins. *J. Mar. Res.* **24**, 82.

RHINES, P. B. 1969 Slow oscillations in an ocean of varying depth. Part I. Abrupt topography. *J. Fluid Mech.* **37**, 161.

WUNSCH, C. 1967 The long period tides. *Rev. Geophys.* **5**, 447.